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Metaphorical motion in mathematical reasoning: Further evidence for pre-motor implementation of structure mapping in abstract domains

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Abstract:

The theory of computation and category theory both employ arrow-based notations that suggest that the basic metaphor "state changes are like motions" plays a fundamental role in all mathematical reasoning involving formal manipulations. If this is correct, structure-mapping inferences implemented by the pre-motor action-planning system can be expected to be involved in solving any mathematics problems not solvable by table look-ups and number-line manipulations alone. Available functional imaging studies of multi-digit arithmetic, algebra, geometry and calculus problem solving are consistent with this expectation.

Keywords: Analogy; Category theory; Computation; Embodied cognition; Event files; Parietal cortex

Introduction

In the dozen years since the publication of *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being* (Lakoff & Núñez, 2000; see also Núñez & Lakoff, 2005), the idea that mathematical reasoning is "embodied" in the sense of being implemented, at least in part, by areas of parietal cortex also involved in the representation of bodily position, orientation and movement has received increasing empirical support. Based on a review of then-available functional imaging results, Dehaene, Piazza, Pinel and Cohen (2003) proposed an influential model of numerical calculation that localized a mental "number line" to the horizontal intraparietal sulcus (HIPS), memory access for numerical symbols to the left angular gyrus (AG) and attentional management for calculation to the posterior superior parietal lobule (SPL). Although not all have focused on the same set of regions of interest, subsequent functional-imaging studies have refined and extended this model for tasks including non-symbolic numerosity judgments (Cantlon, Brannon, Carter & Pelphrey, 2006), arithmetic (Rosenberg-Lee, Lovett & Anderson, 2009; Landgraf, van der Meer & Krueger, 2010), algebra (Qin, Carter, Silk, Stenger, Fissell, Goode & Anderson, 2004; Danker & Anderson, 2007), geometry (Wartenburger, Heekeren, Preusse, Kramer & van der Meer, 2009; Preusse, van der Meer, Deshpande, Krueger & Wartenburger, 2011) and even calculus (Krueger, Spampinato, Pardini, Pajevic, Wood, Weiss, Landgraf & Grafman, 2008). While these studies all demonstrate the involvement of parietal cortex in mathematical reasoning, however, both the roles played by specific areas and the functional relationships between parietal and frontal excitations during mathematical reasoning remain unresolved (Cohen Kadosh & Walsh, 2009; Butterworth, 2010; Piazza, 2010; Anderson, Betts, Ferris & Fincham, 2011).

In contrast to the intensive investigation of the "embodied" aspect of mathematical reasoning, the second major claim of Lakoff and Núñez (2000), that mathematical reasoning is essentially metaphorical, has received less sustained attention. Mowat and Davis (2010) suggested that the mathematical metaphors proposed by Lakoff and Núñez (2000) can be understood as nodes in a larger network of natural-language metaphors, and outlined pedagogical approaches to improve mathematical understanding by decreasing student reliance on metaphors that are not universally applicable. Mowat and Davis (2010) did not discuss the cognitive or neurocognitive implementation of either the specific metaphors proposed by Lakoff and Núñez (2000) or the larger network described as containing them. Independent investigations of the understanding of natural-language metaphors have, however, shown that the processing of novel metaphors in particular activates temporal and parietal areas also involved in motion and action representation (Mashal, Faust, Hendler & Jung-Beeman, 2007; Aziz-Zadeh & Damasio, 2008; Desai, Binder, Conant, Mano & Seidenberg, 2011). While the transition from more "embodied" to more "abstract" representations as metaphors become conventionalized remains to be worked out (Chatterjee, 2010; Schmidt, Kranjec, Cardillo & Chatterjee, 2010; Kiefer & Pulvermüller, 2012), these results are consistent with a representation of mathematical metaphors by cortical areas known to be involved in mathematical reasoning.

Lakoff and Núñez (2000) described mathematics as based on four "grounding metaphors" that define basic arithmetic together with a collection of "linking metaphors" that generate more abstract areas of mathematics from arithmetic. The two primary grounding metaphors are that numbers are like collections of objects and that arithmetic operations are like object-construction operations; these are supplemented by two additional grounding metaphors, that numbers are like lengths and arithmetic operations are like motions along a line. The linking metaphors are more subtle: the first one introduced by Lakoff and Núñez (2000, p. 107 *ff*) is that an algebraic structure is like a Platonic essence. Analytic geometry is introduced, as it is in many classrooms, by the linking metaphor that numbers are like points on a line. This distinction between grounding and linking metaphors in mathematics corresponds, at least in some cases, to the distinction between "groundedness" and "embodiedness" advanced by Fischer (2012): the grounding metaphors of Lakoff and Núñez reflect "universal constraints in cognition that express invariants in the physical world" (Fischer, 2012, p. 162) such as the facts that larger collections of objects really do have more elements and that combining or dividing collections of objects really do change the numbers of elements that they contain. While it is unclear how the notion that an algebraic structure is like a Platonic essence might be "embodied" – aside from in the trivial sense of being implemented by some neurocognitive representation or other – the linking metaphor of numbers being like points on a line appears to be embodied, at least in the case of small (i.e. few-digit) positive integers, in Fischer's sense of "reflect(ing) sensory and/or motor constraints of the human body" (2012, p. 163), in particular the linear, typically left-to-right sweep of visual gaze and the fact that humans have left and right hands with fingers available for counting (Fischer & Brugger, 2011).

Lakoff and Núñez present the distinction between grounding and linking metaphors as corresponding to the distinction between simple arithmetic and "higher" mathematics: algebra, geometry and their more abstract derivatives and extensions. What separates the two in conventional mathematical pedagogy is significant: it is the learning of mathematical facts and rules, often by rote memorization, with the learning of multiplication tables being a canonical example. Lakoff and Núñez discuss multiplication in terms of the intuitive operations of pooling multiple collections of the same size and then counting the number of objects in the pooled collection, and they relate this pooling operation to successive addition, but they do not reflect explicitly on the practice of memorizing times tables. Both lesion and imaging data, however, indicate differences between the implementations of primarily number-line dependent operations such as subtraction and primarily fact-retrieval dependent operations such as multiplication (Dehaene *et al.*, 2003). One can ask, therefore, how the grounding metaphors of Lakoff and Núñez (2000) relate to mathematics not just in theory, but as typically learned and practiced. Relevant to this question is that of how the grounding metaphors are themselves implemented, and whether and how the implementation of the grounding metaphors differs from the implementation of the linking metaphors. Lakoff and Núñez characterize metaphor as a "neurally embodied fundamental cognitive mechanism" (2000, p. 351), but do not address this question of implementation.

The implementation of one of the grounding metaphors of Lakoff and Núñez (2000) is reasonably well understood: considerable data now support the implementation of "numbers are like lengths" – again, at least in the case of small positive integers – by HIPS. The HIPS "number line" appears to support both mental arithmetic and number comparison (Dehaene *et al.*, 2003) as well as non-symbolic numerosity judgments (Cantlon *et al*., 2006). A linear representation of numerical magnitudes extending along a horizontal anatomical axis in close association with motor representations would provide a natural explanation for both the operational momentum effect in number estimation (small numbers are underestimated while large numbers are overestimated; McCrink, Dehaene & Dehaene-Lambertz, 2007; Knops, Viarouge & Dehaene, 2009) and the spatial-numerical association of response codes (SNARC) effect (small numbers are associated with left-hand space while larger numbers are associated with right-hand space; Wood, Willmes, Nuerk & Fischer, 2008), although neither of these effects have been shown to be directly implemented by HIPS. It is unclear, however, whether HIPS serves only as a “mental number line”; Krueger *et al.* (2008) observed HIPS activation when subjects were asked to determine whether integral equations had been solved correctly, an operation sometimes requiring the manipulation of numbers as coefficients or exponents but not obviously involving “number line” reasoning. How the “numbers are like lengths” metaphor is related to the other grounding metaphors of Lakoff and Núñez (2000) is similarly unclear. How, for example, does “numbers are like collections of objects” relate, at the implementation level, to “numbers are like lengths”? The cross-cultural practice of finger counting appears to be the developmental origin of the SNARC effect (Fischer & Brugger, 2011); how are the numbers learned by this method, or by parentally-encouraging counting of objects into or out of other kinds of containers, associated with the linear representation implemented by HIPS? Finally, it remains unclear how the linking metaphors of Lakoff and Núñez (2000) relate to the grounding metaphors. How, for example, do the manipulations represented by mathematical formulas – in the simplest case, the “carry” operations of multi-digit arithmetic – relate at the implementation level to the basic arithmetical operations, and how is this relationship enabled or strengthened by formal instruction?

The present paper advances two hypotheses. The first is that the cognitive processes involved in “higher” mathematics, from the manipulation of formulas in introductory algebra and shapes in basic geometry to more abstract, expert-level manipulations of mathematical objects and relations, can be represented in terms of structure mapping, a general inferential procedure that preserves relational similarities among representations of objects or events and that underlies analogical reasoning (Gentner, 1983; 2003; Markman & Gentner, 2001; Holyoak, 2005). As shown below, the formal structures of two fully-general representations of mathematical reasoning, the classical theory of computation and category theory, both support this hypothesis. Formal structure alone, however, does not determine implementation. The second hypothesis of the present paper is that the diverse results indicating parietal involvement in higher mathematics can be understood within a model in which structure-mapping processes that implement mathematical reasoning are themselves implemented by the pre-motor action-planning system with goal management and attentional control provided by a prefrontal-cingulate loop. A pre-motor implementation of structure mapping has previously been proposed as a model of tool-improvisation capabilities in both humans and non-human animals (Fields, 2011) and of human analogical-reasoning capabilities in domains involving motions and forces (Fields, 2012). In the model proposed here, mathematical reasoning is considered to be an abstraction from reasoning about object manipulation, and hence to involve metaphorically-expressed abstractions of both motion and applied force or effort. This view of mathematical reasoning as implemented by pre-motor structure mapping is consistent with an emerging view of the pre-motor system as a domain-general planning system (Schubotz, 2007; Bubic, von Cramon & Schubotz, 2010) involved ubiquitously in problem solving.

The next section, “Formal representations of structure mapping in mathematics” examines the virtual machine concept developed within the theory of computation and the functor concept developed within category theory. It shows that these fully-general representations of mathematical reasoning can both be viewed as representations of structure mapping as defined by Gentner (1983), although both the virtual machine and functor concepts predate Gentner's work by several decades. The criterion of emulation by a virtual machine and the criterion of commutativity of a mapping diagram relating two or more categories are shown to be strong forms of Gentner's criterion of systematicity for structure mappings. The third section, “Metaphorical motion in formal models of computation” shows that both the theory of computation and category theory are based on the assumption that irreducibly simple motions not only represent but physically implement the most elementary mathematical operations.

The fourth section, “Pre-motor implementation of metaphorical motion” reviews functional-imaging studies of both mathematical reasoning and non-mathematical analogies. It shows that the available data are consistent with a pre-motor implementation of structure mapping in mathematical reasoning, and suggests that a functional-imaging study of category-theoretic “diagram chasing” would contribute to resolving this question. The paper concludes that the semantic analogy capability most studied in analogy research may itself be an analog of a far older, pre-motor capability for motion-based analogical reasoning. It suggests that what is hard about mathematics may not be mathematical reasoning itself, but rather the translation of problems into the motion-based representation that human mathematical cognition appears to employ.

Formal representations of structure mapping in mathematics

Mathematics as implemented by structure mapping

Structure mappings are inferences from one object or event to another object or event that identify and preserve the relational structure shared by the two objects or events (Gentner, 1983; 2003; Markman & Gentner, 2001; Holyoak, 2005). To employ a canonical example, to say that atoms are like the solar system because the orbits of electrons around the nucleus are like the orbits of planets around the sun is to perform a structure mapping (Falkenhainer, Forbus & Gentner, 1989). While Lakoff and Núñez (2000) do not use the term “structure mapping” and do not reference the structure-mapping literature, they present both the grounding and linking metaphors proposed to underlie mathematics, and indeed present “conceptual metaphors” in general, as structure mappings: mappings from a “source” domain to a “target” domain that map objects and relations within the source domain to objects and relations within the target domain in a way that identifies and preserves relational structure (p. 39 *ff*). An explicit representation of the “numbers are like collections of objects” grounding metaphor of Lakoff & Núñez (2000) as a hierarchy of progressively more abstract structure mappings is shown in Fig. 1. These structure mappings preserve the relation “Add 1”; they also preserve other, implicit relational facts common to the four actions shown, for example, the fact that adding another “object” to a “collection” is assumed, in every case, to be without side effects that alter the number or kinds of objects already in the collection. The most basic of these structure mappings (Fig. 1a) simply implements generalization; that such basic structure mappings are non-trivial and hence interesting from an implementation perspective is demonstrated by the fact that they can fail, for example when the “wrong” container for a particular kind of object is rejected by an autistic child. As Lakoff and Núñez (2000) propose that *all* of mathematics rests on the grounding and linking metaphors they present or on others like them, they effectively propose that all of mathematics is implemented, algorithmically, by structure mapping. This is an empirical proposal, and one can ask whether it is true. One can also ask whether a particular implementation of the structure mappings proposed to implement mathematics is suggested by the available data. It is these questions that are considered below.

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Fig. 1 about here.
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A structure mapping is useful to the extent that it is *systematic*, i.e. to the extent that the relations that it preserves are the important, informative and hence potentially inferentially-productive relations within both the source and target domains. “Good” analogies are distinguished from “bad” analogies by systematicity (Gentner, 2003; Holyoak, 2005); “bad” analogies lack systematicity, and are therefore uninformative or misleading. While the systematicity of informal or literary analogies is generally a matter of intuitive conceptual coherence, the systematicity of analogies involving physical forces and motions, such as those involved in improvising tools, is a matter of quantitative scaling; such force-motion analogies are directly tested, and must actually work, when put into practice in the real world (Fields, 2011). Structure mappings may in some cases appear to be systematic, but in fact be – or be discovered to be – not systematic; the canonical “Rutherford” analogy from electron orbits around the nucleus to planetary orbits around the sun is, in fact, not systematic because electrons do not actually orbit the nucleus, and treating them as doing so leads to empirical contradictions (Fields, 2012). The grounding metaphors – or grounding analogies – of arithmetic proposed by Lakoff and Núñez (2000) do not require empirical testing, as they are systematic by definition. In these analogies, the target domains are cognitive constructs in which the *only* relevant relations are those defined by structure mappings from the source domains. Where the grounding metaphors conflict, as in the case of operations involving negative numbers, the conflict is due to dis-analogies between the source domains themselves (Mowat & Davis, 2010).

Recognizing that the statement of a mathematical problem “matches” a known formula and hence can be solved using a known manipulation of that formula clearly involves structure mapping; systematicity is satisfied if the translation of the problem statement to the known formula is in fact correct. For example, if we are told that Alice's house is one km north and three km west of Bob's house, we can calculate the distance from Alice's house to Bob's using the formula $a^2 + b^2 = c^2$ that expresses the Pythagorean theorem. This calculation will be correct provided that Euclidean geometry holds in the world containing Alice's and Bob's houses, and provided that by “distance” we mean shortest straight-line distance. Such translation is relatively easy if the problem is stated using the same formalism employed in the known formula; in this case, structure mapping is syntactic pattern matching. If the problem is stated in a different formalism, or in natural language as with the example above or the notorious “word problems” of school arithmetic, performing such structure mappings correctly can be considerably more difficult. Most students learn long division, basic algebra and beginning calculus by learning and applying formulas. As mathematical reasoning becomes more expert, however, it relies less on learned formulas and looks less like syntactic symbol manipulation; an artfully-constructed proof is a conceptual exercise in which the chosen syntax plays the role it plays in an artfully-constructed poem. This is, indeed, the point that Lakoff and Núñez (2000) are making in their discussion of the concepts underlying Euler's famous formula $e^{i\pi} = -1$.

The notion of mappings between structures is, of course, itself a mathematical notion; the centrality of the concepts of “structure” and “mapping” within mathematics by itself lends credence to the idea that mathematics is a network of structure mappings. The remainder of this section considers two formalizations of the notion of structure mapping within expert-level mathematics, both of which incorporate explicit definitions of systematicity, and both of which significantly predate the characterization of structure mapping as an algorithm for analogical reasoning by Gentner (1983).

Virtual machines and emulation

The classical theory of computation rests on two foundations: a set of strictly-equivalent formal models of computation exemplified by the Turing machine, and a universality claim, the Church-Turing thesis, that states that any process that intuitively “counts” as computation can be represented as a computation using any one of the formal models (Turing, 1937; Hopcroft & Ullman, 1979 is a standard reference; Galton, 2006 reviews recent controversies surrounding the Church-Turing thesis). One implication of the Church-Turing thesis is that any intuitively "computational" process that can be implemented on any one formal model of computation can also be implemented on any other formal model of computation. It is this implication of the Church-Turing thesis that allows programs written using different formal models of computation – i.e. different programming languages – to be regarded as alternative implementations of the "same" computation; the Church-Turing thesis is thus crucial to the practice of programming.

Early computers were programmed by interacting directly with the hardware. The development of programming languages interposed a layer of software between the programmer and the hardware, allowing programs to be written in a way that was at least approximately hardware-independent. Since the early 1970s it has become commonplace to interpose sufficient software between a computer's hardware and the software tools employed by application programmers that programs can be made fully hardware-independent, i.e. portable. Such interposed software constitutes a *virtual machine* – a specified software system that a programmer can treat as if it were a physical machine when writing application software (Goldberg, 1974; Tanenbaum, 1976). The virtual machine "model" of computation frees programmers from having to know anything about the hardware on which their programs will run; it renders the Church-Turing thesis true *in practice* for programmers. From the perspective of a computer's users, application software packages can themselves be considered virtual machines. A laptop running a web browser, for example, can be considered to be a web-browsing machine; the user needs not know anything about what the laptop is doing at the hardware, the operating-system, or even the application-software level to successfully browse the web. Users of "cloud" applications on the internet need not know what kind of computer is running the application, or even where it is located; from the user's perspective, it is as if a distinct, specialized computing machine implemented each of their applications.

A "virtual" process – such as web browsing – running on a virtual machine is related to the physical, i.e. electronic or possibly optical process running on a computer's hardware by *emulation*: the programmed hardware behaves *as if* it were a specialized machine constructed to execute the virtual process. Successful emulation serves as a criterion for the correctness of a virtual-machine implementation. One can, therefore, represent the mapping from a physical process occurring in the hardware to a virtual process occurring in a virtual machine implemented by the hardware as a structure mapping in which systematicity is strictly defined by the criterion of emulation; Fig. 2 shows such a representation. In this representation, a non-systematic structure mapping between hardware and virtual machine is a program with bugs – a program that does not, in fact, do what it is supposed to do. If viewed in reverse, a systematic emulation mapping defines a correct semantic interpretation of the behavior of the hardware as the behavior of the virtual machine, i.e. as a coherent, intuitively-meaningful computation. If the hardware is an internally-uncharacterized "black box" or a naturally-occurring system such as a brain, an independently-verified emulation mapping may be viewed as an explanation of the behavior of the hardware (Marr, 1982; Cummins, 1983). From this perspective, the "grounding metaphors" of Lakoff and Núñez (2000) can be considered to be statements of emulation relationships between abstract mathematical processes and biochemical or bioelectrical processes in brains that are proposed as explanations of human mathematical-reasoning capabilities.

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Fig. 2 about here
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By allowing any computational process **P** executed by a given hardware system to be considered to be equivalent to an execution of **P** on any formal model of computation, the Church-Turing thesis allows the emulation relationship between hardware and virtual machine to be considered as a special case of a general emulation relationship between two virtual machines, i.e. between two purely formal, mathematical entities. This general relationship defines a precise sense, representable as in Fig. 2, in which one mathematical process is "like" another, or can be regarded as an "interpretation" or "model" of another. This generalized sense of emulation provides the basis for the "denotational" approach to programming-language semantics (Stoy, 1977). From this perspective, programs are "models" or "implementations" of computational processes in the precise sense of being structure-mapping analogies of those processes with emulation as the criterion of systematicity.

Categories, functors and diagram commutativity

The mathematics of category theory generalizes the methods employed in abstract algebra and algebraic topology to investigate the relationships between different mathematical systems (Eilenberg & Mac Lane, 1945; Mac Lane, 1972 is a standard reference; Adámek, Herrlich & Strecker, 2004 is an accessible introduction). A mathematical *category* is a collection of objects and a collection of arrows (or "morphisms") such that: (1) for each pair of objects A and B there is at least one arrow $f: A \rightarrow B$ and for each object A there is an "identity" arrow $i_A: A \rightarrow A$; (2) any two arrows $f: A \rightarrow B$ and $g: B \rightarrow C$ can be composed (indicated by "∘" and read "following") to form a new arrow $g \circ f: A \rightarrow C$; and (3) this process of composing arrows satisfies the associative law of conventional arithmetic. Conventional set theory and hence all of conventional mathematics can be represented in terms of objects and arrows within category theory.

Categories are related by functors. A *functor* is a family of functions that map both the objects and the arrows of some category **A** to the objects and arrows of another category **B** in a way that preserves identities and arrow composition. The action of a functor F: $\mathbf{A} \rightarrow \mathbf{B}$ is conventionally illustrated by a diagram such as shown in Fig. 3. A diagram in which all paths from one object to another are equivalent is *commutative*; for example, if $f \circ F = F \circ g$ for every f in **A** and g in **B** in Fig. 3, the diagram commutes. If two categories can be related by one or more functors that yield commutative diagrams, they share potentially inferentially-productive structural commonalities; categories that cannot be related by diagrams that commute lack such commonalities. Mapping the real numbers with addition as an operation to the real numbers with multiplication as an operation, for example, is inferentially productive: it reveals that zero and one are functionally analogous as identity elements, and gives rise to the mathematics of exponents and logarithms. Many of the questions addressed by category theory turn on the issue of whether a diagram relating categories commutes; hence "diagram chasing" is a preferred and often straightforward category-theoretic proof technique.

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Fig. 3 about here.
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The diagrams shown in Fig. 3 are transparently structure mappings in which systematicity is assured if the diagrams commute. Indeed, category theory can be viewed as a mathematical theory of structure mapping, where the structures in question represent mathematical systems in the most general possible way. Viewed in this way, what category theory tells us is that the relationships between mathematical systems that "make sense" are the ones that can be represented as structure mappings with diagram commutativity as the criterion of systematicity. Mathematical systems that cannot be related by commutative diagrams are "bad analogies" for each other: they do not share interesting structure, and working between them is unlikely to be inferentially productive. As in the case of the Church-Turing thesis, this is a claim about how our mathematical intuitions – how what counts as "interesting" in mathematics – can be captured within a formalism.

The virtual-machine concept and the programming languages that support it, category theory with its commutative diagrams, Feynman diagrams representing interactions in particle physics, and flow charts representing the transfer of information or control between components of a complex system all have the same underlying formal structure (Baez & Stay, 2011) and were all developed in the mid-20th century, at a time when it became both conceptually and technologically possible and economically and militarily necessary to perform mathematical calculations of a previously-infeasible complexity. These forms of diagrammatic calculation all employ strict criteria of systematicity; while these criteria are implicit in Feynman diagrams and flow charts, they can be made explicit by considering either of these representations as depicting virtual machines. While diagrammatic reasoning is a relatively new research field within cognitive psychology (Larkin & Simon, 1987; Richards, 2002; Gottfried, 2012), several generations of mathematicians, computer scientists, physicists and engineers have demonstrated that diagrammatic representations of complex computations are far easier to understand and employ to obtain practical results than are their purely-linguistic equivalents. As Larkin and Simon (1987) emphasize, a diagrammatic presentation typically reduces the search time required to solve a problem, and solutions to problems presented by diagrams can sometimes be recognized in the course of understanding the diagram; "diagram chasing" as a solution strategy exploits this fact. The next two sections outline a model of why this should be so.

Metaphorical motion in formal models of computation

The arrows in Figs. 2 and 3 are enough to suggest that *motion* plays a significant role in diagram-driven mathematical reasoning. Moving something from one place to another changes its state (in particular, its location); it also changes both the context originally occupied by the object and the context into which the object is introduced. Object motion thus *implements* state change both for objects and for contexts involving multiple objects. The "numbers are like collections of objects" grounding metaphor of Lakoff and Núñez (2000) exploits this fact: moving an object from one collection to another is both a physical and a numerical operation.

The arrows in Figs. 2 and 3 also, however, capture a second deep metaphor: the locational state change physically implemented by a motion can be employed to represent any state change that can be explicitly imagined as an event. Lakoff and Johnson (1999) call the representation of general processes by motions the "event-structure metaphor"; Lakoff and Núñez (2000) employ it in their discussion of infinity. The diagrammatic representations employed in computing theory and category theory suggest that this metaphor is in fact a grounding metaphor for all of mathematics, one that is both logically and architecturally even more basic than "numbers are like collections of objects." They suggest, in other words, that "state changes are like motions" is the fundamental metaphor driving mathematics.

Motions are able to represent processes because motions *implement* processes; this is an "invariant in the physical world" and hence confers "groundedness" on such representations in Fischer's (2012) sense. Lakoff and Núñez (2000) emphasize the simplicity of the motions implementing such developmentally-early arithmetic processes as counting on the fingers or moving objects into and out of some designated space. The notion that computational processes can be broken down into simple motions is generalized by the Turing machine, in which the irreducibly simple motions of marking, erasing, and moving along the tape are made to represent all possible computations. Most contemporary computers involve a similar generalization, again guided by the Church-Turing thesis, that substitutes the flow of electrical current for the motion of discrete objects. The somewhat pejorative notion that computation is “just pushing symbols around” – a notion made famous by Searle's (1980) “Chinese Room” argument against artificial intelligence – expresses the same generalization. From this perspective, the syntactic operations involved in manipulating formulas can be considered to be state transitions within a space of language-like formal representations that are implemented by a specified set of simple motions of well-defined symbols.

The simplicity of elementary mathematical operations, and indeed the goal of discovering small sets of irreducibly simple operations that are arbitrarily inferentially productive within a chosen domain are distinguishing features of mathematics (and mathematical physics) as cognitive activities. Irreducibly simple mathematical operations are meant to be *obvious*; as algebraic topologist Burt Casler (1927-2012) was fond of saying, a mathematician's job is “to turn impossible problems into trivial problems.” In both the theory of computation and category theory, the irreducibly simple operations can be viewed as irreducibly simple motions: marking, erasing or moving along the tape, or simply following an arrow from one object to another. The centrality of this idea of an irreducibly simple motion to mathematics is reflected in the centrality of arrow composition as the sole operation assumed within category theory. If any arrow can be regarded, at least notionally, as a composition of arrows representing irreducibly simple motions, then the theory needs not concern itself with what the arrows *are*, and hence can focus exclusively on how arrows can be sequentially combined. The theory similarly needs no assumptions about what the objects are; it is enough that they are whatever the manipulations represented by the arrows manipulate. Indeed as Adámek, Herrlich and Strecker (2004) emphasize early in their presentation (p. 42, Remark 3.55), category theory can be fully formulated with no notion of “object” at all. It is, therefore, not incorrect to claim that from the idea of an irreducibly simple motion and the notion that such motions can be sequentially composed, all of mathematics follows. That the number of distinct arrows required to formulate a non-trivial object-free category theory – three – is the same as the number of elementary operations of a Turing machine serves to emphasize this conclusion.

Pre-motor implementation of metaphorical motion

Hypothesis: Structure mapping in mathematics is implemented by event-file manipulation

In the closing of their paper, Larkin and Simon (1987) speculated that diagrammatic reasoning in which the diagrams were merely imagined would be essentially equivalent to diagrammatic reasoning using external, observed diagrams. Subsequent experimental investigation of both visual imagination (Kosslyn, Thompson & Ganis, 2006; Moulton, S. T. & Kosslyn, 2009) and episodic-memory retrieval (Moscovitch, 2008; Ranganath, 2010) has confirmed this speculation: up to modulation by the rostral prefrontal “reality monitoring” system (Simons, Henson, Gilbert & Fletcher, 2008), perception and imagination activate the same object and motion representations in temporal and parietal cortex. As representations of objects moving within a context are representations of events, Hommel (2004) has termed such activations "event files." To the extent that mathematical reasoning involves either perceived or explicitly-imagined sequences of motions or manipulations, one would expect such reasoning to be implemented by sequential processing of event files.

It has been shown previously that structure-mapping analogies involving forces and motions that are performed in the course of tool improvisation (Fields, 2011) or even abstract physical reasoning (Fields, 2012) can be represented as manipulations of event files implemented by the pre-motor action-planning system. In both of these cases, structure mapping results in the construction of a novel action-sequence representation that combines components of previous action sequences, with the forces applied scaled to produce the motions required. The action plans constructed by tool-improvisation structure mappings represent the body as actor and are executable as tool-use actions. The action plans constructed by physical-reasoning structure mappings employ the body as a metaphorical representation of some external object, and constitute predictions of that object's future behavior. These latter structure mappings are thus critically dependent on the ability of the mirror system, in particular mirror components of SPL (Nassi & Callaway, 2009), to respond to motions of observed or imagined inanimate objects (Schubotz & von Cramon, 2004; Engel, Burke, Fiehler, Bien & Rosler, 2007; Catmur, Walsh & Heyes, 2007; Heyes, 2010; 2012) and hence to represent the motions of such objects as force-carrying actions.

While the formal structures of both the theory of computation and category theory support Lakoff and Núñez's (2000) contention that *all* mathematical reasoning is based on metaphorical motion and hence implementable by structure mapping, reflection on such phenomena as the memorization of times tables suggests that this possibility is not realized as a matter of fact: *some* mathematical reasoning, such as the inference from '2 · 2' to '4', appears to be implemented, at least in many individuals, by table lookup. The HIPS number line appears, moreover, to serve as an at least somewhat specialized processor of numerical order and magnitude, at least for few-digit positive integers. One can hypothesize, however, that the *rest of* mathematical reasoning, including formula manipulation, geometric reasoning, and the more abstract reasoning of expert-level mathematics, is implemented by the same kinds of pre-motor structure-mapping processes that appear to implement tool-improvisation and physical-reasoning analogies. If this hypothesis is correct, the primary role of memory retrieval in mathematical reasoning is similar to its primary role in physical reasoning: it provides facts, often in the form of templates such as abstract diagrams or formulas, for input into structure-mapping processes. That this hypothesis is at least plausible is suggested by the very similar usage of diagrams in mathematical and physical reasoning and by the structural similarities between the diagrams employed. Indeed, a single neurocognitive mechanism for both mathematical and physical reasoning would go a considerable way toward explaining the "unreasonable effectiveness of mathematics in the natural sciences" emphasized by Wigner (1960).

If structure mapping in mathematical reasoning is implemented by the pre-motor action planning system, direct neurofunctional assays such as fMRI or transcranial magnetic stimulation (TMS) would be expected to reveal specific activations in anatomically-distinct components of this system during mathematical reasoning or problem solving. Specifically, one would expect manipulations of mathematical entities to elicit posterior parietal and possibly supplementary motor cortex activations representing imagined motions, and one would expect requirements for syntactically-complex re-representations of problems to match known formulas or other reasoning templates to elicit inferior frontal activations, particularly in areas of inferior frontal gyrus (IFG) known to be involved in complex syntactic processing (Santi & Grodzinsky, 2007; Bahlmann, Schubotz & Friederici, 2008;

Makuuchi, Bahlmann, Anwander & Friederici, 2009; see Fedorenko, Nieto-Castañón & Kanwisher, 2012 for a discussion of individual differences). One would expect activity in either area to scale with problem demands. One would, however, expect *subjective* reports of difficulty to correlate not with increased posterior-parietal or supplementary-motor activation but with increased activations of inferior frontal and prefrontal cortex (PFC), especially dorsolateral PFC for problems involving higher working-memory demands and rostral PFC for problems requiring multi-tasking or the suppression of distractions.

Results from functional imaging studies of mathematical problem solving

The broadly-confirmed greater involvement of SPL in arithmetical operations as the requirement for computation increases relative to the requirement for retrieval of results from memory (Dehaene *et al*., 2003; Rosenberg-Lee, Lovett & Anderson, 2009; Rosenberg-Lee, Chang, Young, Wu & Menon, 2011) is consistent with the present model, particularly if the "rules" being followed are represented as formal templates (i.e. formulas with variables or "slots") that must be actively imagined. It is interesting that SPL activation does not respond differentially as the difficulty of multiplication problems increases from one-digit problems to two-digit problems, while supplementary motor cortex and dorsolateral PFC activations do respond differentially (Landgraf, van der Meer & Krueger, 2010). The representation of a single "carry" operation may be insufficient to generate differential SPL activation; a test of this effect with problems involving more digits and hence presumably imposing a greater demand on imaginative resources would be interesting. The lack of differential activation of AG in this data set similarly suggests that the difference between one- and two-digit multiplication problems is insufficient to reveal differential demands on number-memory retrieval. Rosenberg-Lee *et al*. (2011, Fig. 2) report greater activation of both inferior frontal cortex and dorsolateral PFC in multi-step division compared to other arithmetic operations, but they do not consider these areas as regions of interest and do not discuss this differential activation. It is, however, consistent with the greater working-memory requirements of the interleaved multiplication, subtraction and carry operations of long division.

Studies of algebra (Qin *et al.*, 2004; Danker & Anderson, 2007), geometry (Wartenburger *et al.*, 2009; Preusse, van der Meer, Ullwer, Brucks, Krueger & Wartenburger, 2010; Preusse *et al.*, 2011) and calculus (Krueger *et al.*, 2008) provide more sensitive tests of the current model. Solving simple algebra problems (e.g. 3x + 4 = 19) requires manipulations that are both described and routinely taught using spatial metaphors (e.g. an equation has "sides" separated by the '=' sign) suitable to the linear form in which such equations are standardly written. Qin *et al.* (2004) demonstrate contemporaneous left parietal, anterior cingulate and prefrontal activity in algebraic equation solving; Danker and Anderson (2007) confirm these activations. Both interpret their data in terms of the adaptive control of thought – rational (ACT-R) model, which represents parietal activations as performing data manipulation, cingulate activations as managing goals and prefrontal activations as performing rule and fact retrieval from memory (Anderson, 2005). Although ACT-R is not described in terms of structure mapping, the "unwind" operation that forms the core of the ACT-R equation solver is effectively a structure-mapping operation that sequentially re-represents parsed equation components using a small number of formal templates; a parietal implementation of "unwind" is, therefore, a parietal implementation of structure mapping. Neither Qin *et al.* (2004) or Danker and Anderson (2007) report significant HIPS activation, consistent with primarily formula-based as opposed to primarily numerical operations in algebraic problem solving.

Structure mapping is examined explicitly in the geometric-analogy studies of Wartenburger *et al.* (2009) and Preusse *et al.* (2010; 2011), which demonstrate coordinated parietal and frontal activity modulated by task difficulty. Both inferior (IPL) and superior parietal lobules and supplementary motor cortex are active in geometric analogies, with a bias to the left hemisphere as expected. Interestingly, Preusse *et al.* (2011) report greater parietal activity relative to frontal activity in individuals with higher fluid intelligence (Cattell, 1971), suggesting that such individuals require either fewer attentional-control resources or less working memory (Gray, Chabris & Braver, 2003) to solve geometry problems than individuals with lower fluid intelligence. Preusse *et al.* (2011) also report increasing activation of left IFG specifically with increased problem difficulty, irrespective of fluid intelligence score, suggesting that the demand for syntactic manipulations – in the stimuli employed by Preusse *et al.* (2011), manipulations of object shape – is independent of fluid intelligence. As with studies of algebra problem solving, these studies of geometric problem solving do not reveal significant HIPS activations, consistent with a specialized role of HIPS in number representation. The study of function integration, another template-driven computational process, by Krueger *et al.* (2008) demonstrates differential SPL but not IPL activity, similar to results for algebraic problem solving. A greater requirement for multi-step sequential reasoning, and hence greater demand for working memory, in solving integrals compared to simple algebra problems or geometric analogies is reflected by significant activation of left dorsolateral PFC. The significant activation of HIPS by function integration problems is surprising, as only two of the seven integrals presented as examples (Krueger *et al.*, 2008, Fig. 1) contain numbers in addition to symbols for operators (e.g. '*ln*') or variables (e.g. '*x*'). As noted earlier, this observation may indicate that HIPS has functions other than number representation; such additional functions may be recruited by the manipulations of operator symbols required to solve integration problems, but not required to solve either the algebra or geometry problems considered here.

Where is the mapping step in mathematical structure mapping implemented?

The consistent activation of parietal areas in the studies reviewed above indicates, and is broadly interpreted as indicating, parietal involvement in the representation of mathematical structures such as equations, diagrams or geometrical figures. The consistent activation of prefrontal areas indicates, and is broadly interpreted as indicating, prefrontal involvement in the goal-dependent control of mathematical reasoning. Rostral PFC appears to be critical, in particular, to the management of multiple processing streams and the prevention of interference between streams, for example when a partial result must be “set aside” prior to integration with a later result (De Pisapia & Braver, 2008), and to the extension or modification of known rules to make them applicable to novel problems (Anderson *et al*., 2011). These studies, however, leave open the key question of where and how the transformation of one mathematical structure into another is implemented. This is the *mapping* step within structure mapping. The event-file manipulation model (Fields, 2011, 2012) of structure mapping represents this mapping step as modulated or even controlled by prefrontal activity, but as implemented by parietal and posterior-frontal components of the pre-motor action planning system. No direct functional imaging evidence for a pre-motor implementation of the mapping step in mathematical structure mappings is available. Before considering the indirect evidence supporting a pre-motor implementation, however, it is necessary to evaluate the most plausible alternative, that the mapping step in mathematical reasoning is implemented by prefrontal cortex.

Analogy has traditionally been considered to be a human-specific component of “central cognition” – to be “why we're so smart” compared to other species (Gentner, 2003; see Fleming, Beran, Thompson, Kleider & Washburn, 2008; Kennedy & Fragaszy, 2008; Fields, 2011 for opposing views). Functional imaging studies of analogical reasoning have consistently demonstrated activation of rostral PFC by the mapping step, and have proposed a prefrontal implementation of mapping (Bunge, Wendelken, Badre & Wagner, 2005; Green, Fugelsang, Kraemer, Shamosh & Dunbar, 2006; Wendelken, Nakhabenko, Donohue, Carter, C & Bunge, 2008; Knowlton & Holyoak , 2009; Cho, Moody, Fernandino, Mumford, Poldrack, Cannon, Knowlton & Holyoak, 2010; Krawczyk, McClelland & Donovan, 2010; Volle, Gilbert, Benoit & Burgess, 2010; Watson & Chatterjee, 2012; reviewed by Krawczyk, 2012). While the types of analogies studied include both word-based (Bunge *et al.*, 2005; Green *et al.*, 2006; Wendelken *et al.*, 2008) and picture-based (Cho *et al.*, 2010; Krawczyk, McClelland & Donovan, 2010) semantic analogies and spatial-layout analogies (Volle *et al.*, 2010; Watson & Chatterjee, 2012), however, analogies specifically involving relations between motions have not been directly investigated by functional imaging. Whether the mapping step in *motion-based* analogies is implemented by PFC is, therefore, left open by existing experimental studies.

Three lines of reasoning suggest a posterior-parietal implementation of mapping in motion-based analogies. First, the posterior-parietal cortex provides the representation of motion employed for motor planning, a task that requires the representation of motions at multiple levels of abstraction. Unlike for spatial layouts, for example, a distinct, more abstract but still precise representation of motion patterns is not available; natural language in particular is very poor as a motion representation. If motions are re-represented as amodal abstractions by PFC prior to a PFC-implemented mapping step, it is entirely unclear how they would be re-represented. Second, structure mapping provides a natural description of the inferences employed to plan novel motions. As discussed earlier, motion planning requires assembling new relational structures from previously-employed relational structures subject to the strong systematicity constraint of quantitatively-correct force-motion scaling. The facility of non-human animals in planning novel motions argues strongly against any requirement for a language-dependent or similarly abstract or amodal re-representation of motions to enable mapping across motion representations. The bilateral IFG activations observed in the studies of spatial-layout analogies by Volle *et al.* (2010) and Watson and Chatterjee (2012) are interesting in this regard. The study of Volle *et al.* (2010), which employed sequences of letters as layouts, reported bilateral opercular IFG (Brodmann Area 44) activation but only right triangular IFG (Brodmann Area 45) activation (Table 1), while the study of Watson and Chatterjee (2012), which employed arrays of geometrical symbols as layouts, reported bilateral triangular but only right opercular IFG activation (Fig. 3C). Watson and Chatterjee (2012) interpret their observed opercular IFG activation in terms of incorrect-response inhibition, a form of cognitive control. Volle *et al.* (2010) do not discuss their observed IFG activations, but they are consistent with the use of spatially-arranged letter combinations as stimuli. These results for spatial-layout analogies, which both Volle *et al.* (2010) and Watson and Chatterjee (2012) interpret as indicating the implementation of mapping by rostral PFC, can be contrasted with those of the geometric-analogy study of Pruess *et al.* (2011), which reports increasing left IFG activity with increasing problem difficulty as discussed above. Finally, the substantial facilitation of both mathematical and physical reasoning provided by diagrams depicting metaphorical motion, again as discussed previously, suggests that structure mappings defined over diagrams are less cognitively demanding than structure mappings defined over language-based representations, which the work of Bunge *et al.* (2005), Green *et al.* (2006) and Wendelken *et al.* *(*2008) shows is implemented by PFC.

While diagrammatic reasoning in many domains could be employed to evaluate the potential roles of posterior parietal cortex and PFC in the mapping step using dissociation designs such as those of Cho *et al.* (2010), Krawczyk, McClelland and Donovan (2010) or Watson and Chatterjee (2012), “diagram chasing” in category theory appears to provide a particularly pure case of diagram-based structure mapping for which examples with an extraordinary range of difficulty, an alternative, language-based notation with provably identical mathematical content and a population of experts – advanced mathematics students – are readily available. An experimental analysis of category-theoretic diagram chasing as a proof technique would, moreover, be an experimental analysis of one of the most general forms of mathematical reasoning yet devised.

Conclusion

Expert-level mathematics is one of the most abstract cognitive activities in which human beings engage. The idea that such an abstract activity could be described as both “embodied” and “metaphorical” was highly controversial when Lakoff and Núñez (2000) proposed it; Dehaene (2011) could still claim, a decade of intensive work on embodied cognition later, that “most mathematicians, overtly or covertly, are Platonists” (p. xi) and hence unlikely to support either embodiment or metaphor as characterizations of their thinking. The very notation, however, of much expert-level mathematics suggests that the metaphor “state changes are like motions” underlies mathematical cognition. The available data on the human implementation of mathematical reasoning is consistent with the hypothesis that, with the exception of problems solved exclusively by memory look-up or number-line manipulation, posterior-parietal and supplementary-motor motion representations in fact underlie not just some but all mathematical cognition.

The empirical plausibility of a pre-motor implementation of structure mapping not just in mathematics but also in tool improvisation (Fields, 2011) and physical reasoning (Fields, 2012) suggests that pre-motor structure mapping is both ancient and highly representationally plastic. If this is the case, a second hypothesis is suggested: that prefrontal implementations of structure mapping are learned generalizations of pre-motor structure mapping. Were this to be the case, the semantic analogies upon which most analogy research has focused to date would be evolutionarily – and perhaps also culturally – late analogs of much older analogies, those concerned with perceptible forces and motions. Understanding the implementation of these older, pre-motor analogies may shed new light on both the implementation and the cognitive semantics of their prefrontal descendants.

The present conceptualization of mathematical reasoning as essentially analogical also raises a question about mathematics itself: why is mathematics so hard? If mathematics is based on a few simple motions – as the Turing machine shows that it can be – why are so few human beings adept at mathematics? One possible answer is that more human reasoning may be “mathematical” than generally counts as such: the largely-unconscious cognition that humans employ to move their bodies through ordinary life may be productively conceptualized as “mathematical” cognition. From this perspective, a mathematical pedagogy that emphasized metaphorical motions, either by bodily movements or diagrams, might facilitate the teaching of mathematics, as some recent pedagogical studies in fact suggest (Kellman, Massey & Son, 2009; Radford, 2009; Blair & Schwartz, 2012). Another answer is suggested by the practice of programming. What is hard about programming is the translation of a statement of desired behavior – a statement that may be far vaguer than a “problem specification” – into the simple operations stipulated by a programming language. Such decomposition of a problem stated in everyday or even specialized terms into an abstract vocabulary is itself a form of structure mapping, one that often requires the recognition of relational similarities across large semantic distances. Mathematics may be a natural human skill; abstract problem decomposition, however, decidedly is not.

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The author declares that he has no financial or other conflicts of interest with regard to the research reported here.

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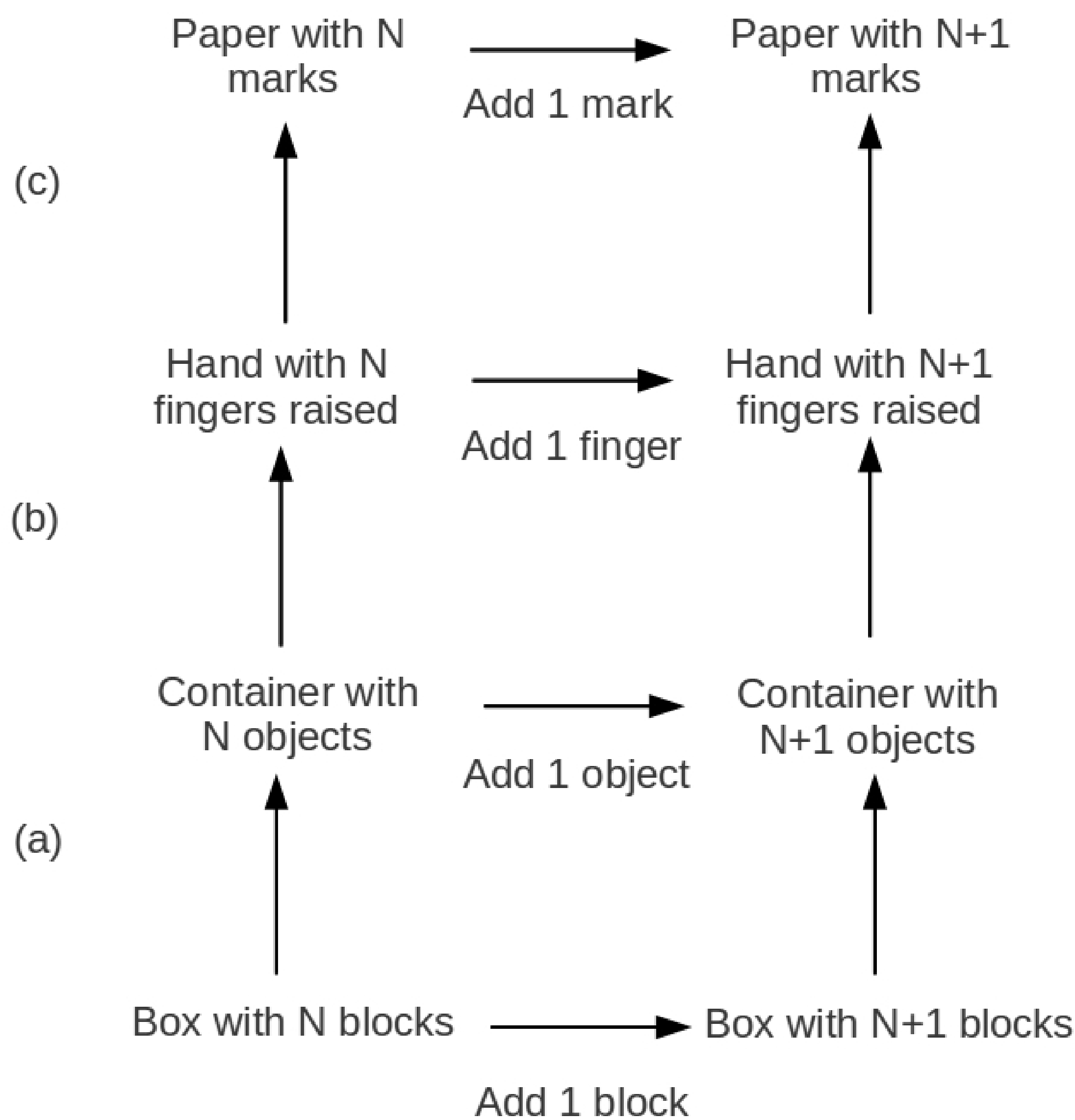
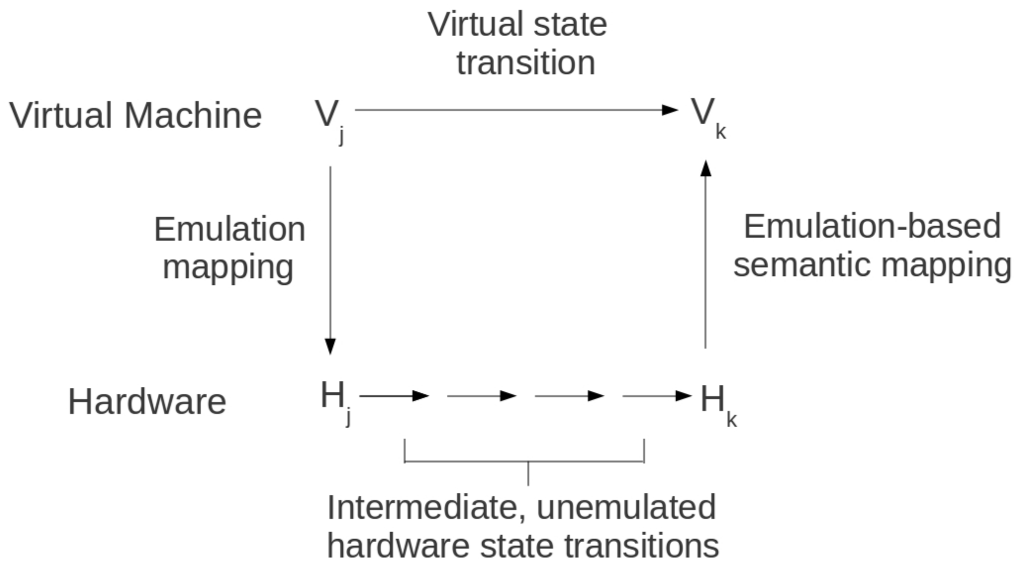
Figure Captions

Fig. 1: Representation of the "numbers are like collections of objects" grounding metaphor of Lakoff & Núñez (2000) as a hierarchy of progressively more abstract structure mappings (vertical arrows). (a) Structure mapping from the act of adding a specific object to a specific container to the act of adding an arbitrary object to an arbitrary container. The "Add 1" relation is preserved. (b) Structure mapping from the act of adding an object to a container to the act of raising a finger. The "Add 1" relation is preserved. (c) Structure mapping from the act of raising a finger to the act of marking a paper. The "Add 1" relation is preserved. Composing mappings (a) – (c) relates the accumulation of blocks in a box to the accumulation of marks on a paper.

Fig. 2: Representation of emulation as structure mapping. Systematicity is defined as correctness of emulation. Viewed in reverse, the emulation mapping defines a semantic interpretation of the behavior of the hardware. Note that there is no assumption that an emulation map or an emulation-based semantics is defined for every state of the hardware. The diagram can be generalized to represent an emulation mapping between virtual machines.

Fig. 3: (a) Diagram illustrating a functor *F* mapping a category **B** to another category **A**. The diagram *commutes* in case $f \circ F = F \circ g$ for every arrow *f* in **A** and *g* in **B** for which the composition of arrow and functor is defined. (b) An instance of (a) showing specific arrows *r* and *s* defined for specific objects $\mathbf{a}_i$ and $\mathbf{a}_j$ of **A** and $\mathbf{b}_k$ and $\mathbf{b}_l$ of **B**, respectively. Commutativity requires that $r \circ F = F \circ s$.


Paper with N marks
Add 1 mark
Paper with N+1 marks
(c)
Hand with N fingers raised
Add 1 finger
Hand with N+1 fingers raised
(b)
Container with N objects
Add 1 object
Container with N+1 objects
(a)
Box with N blocks
Add 1 block
Box with N+1 blocks


Virtual state
transition
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Vj
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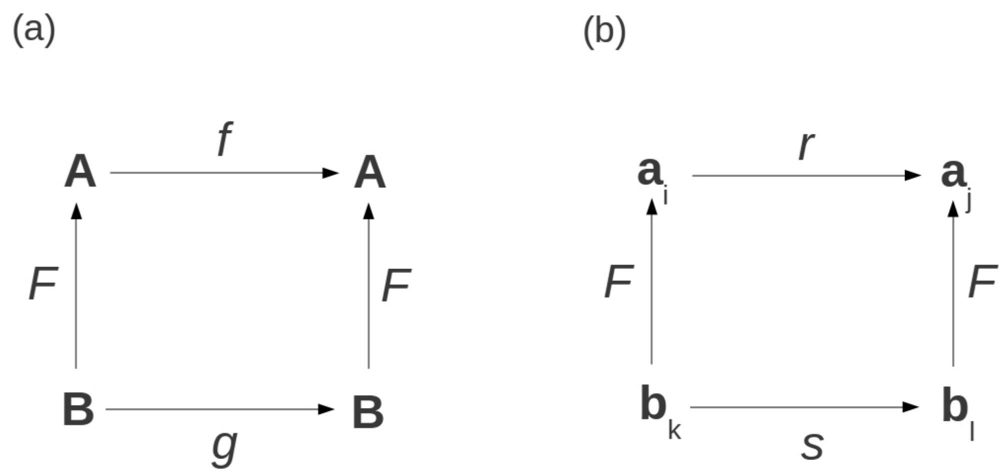
Emulation
mapping
Emulation-based
semantic mapping
Hardware
Hj
Hk
Intermediate, unemulated
hardware state transitions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